On the number of transversals and multiplexes in iterated quasigroups

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Definitions

A pair $G = (X, *)$ is a binary quasigroup of order n if $|X| = n$ and

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A latin square of order n is an $n \times n$ table filled by n symbols so that each line contains all n symbols.

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A *d-ary quasigroup f of order n* is a d-ary operation on a set X of size n such that the equation $f(x_1, \ldots, x_d) = x_0$ has a unique solution for any one variable if all other d variables are specified arbitrarily.

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A 3-dimensional latin hypercube of order 4

$$\left|\begin{array}{cccc|cccc|cccc|cccc} 0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 & 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 2 & 3 & 0 & 1 \\ 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 & 0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 & 2 & 3 & 0 & 1 & 1 & 0 & 3 & 2 & 0 & 1 & 2 & 3 \end{array}\right|$$

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2	3	0	1	3	2	1	0	0	1	2	3	1	0	3	2
3	2	1	0	2	3	0	1	1	0	3	2	0	1	2	3

If $I_n^d = \{(\alpha_1, \ldots, \alpha_d) : \alpha_i \in \{0, \ldots, n-1\}\}$ then a *d-dimensional matrix A of order n* is an array $(a_\alpha)_{\alpha \in I_n^d}$, $a_\alpha \in \mathbb{R}$. A *line* is a 1-dimensional submatrix of A, and a *hyperplane* is a $(d-1)$-dimensional submatrix. A *d-dimensional latin hypercube of order n* is a multidimensional matrix filled by n symbols so that each line contains all different symbols.

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Given a binary quasigroup $G = (X, *)$ of order n, the d-iterated quasigroup $G^{[d]}$ is a d-ary quasigroup $f^{[d]}(x_1, \ldots, x_d)$ of order n such that

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The Cayley table of the 3-iterated 3-ary quasigroup $G^{[3]}$

0	1	2	3	1	0	3	2	2	3	0	1	3	2	1	0
1	0	3	2	0	1	2	3	3	2	1	0	2	3	0	1
2	3	0	1	3	2	1	0	0	1	2	3	1	0	3	2
3	2	1	0	2	3	0	1	1	0	3	2	0	1	2	3

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3	2	1	**0**	2	3	0	1	1	0	3	2	0	1	2	**3**

Trivial upper bound on the number of k-multiplexes: $\left(\frac{(kn)!}{k!^n}\right)^d$.

Motivation

Proposition

If k is odd and d and n are even then there exist d-dimensional latin hypercubes of order n with no k-multiplexes.

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Consider the Cayley table Q of the d-iterated group $\mathbb{Z}_n^{[d]}$

$$q_\alpha = \alpha_1 + \ldots + \alpha_d \mod n;$$

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$$\sum_{i=1}^{kn} q_{\alpha^i} = k\frac{n(n-1)}{2} \not\equiv 0 \mod n.$$

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$$\sum_{i=1}^{kn}(\alpha_1^i + \ldots + \alpha_d^i) = dk\frac{n(n-1)}{2} \equiv 0 \mod n.$$

Motivation

Theorem (T., 2015; Glebov, Luria, 2016)

The maximum number of transversals in d-dimensional latin hypercubes of order n is asymptotically equal to

$$\frac{n!^{d-1}}{e^n}(1+o(1))^n \text{ as } n \to \infty.$$

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Theorem (Eberhard, 2017+)

If G is an abelian group of order n and $(d+1)\sum_{g \in G} g = 0$ then the number of transversals in the d-iterated group $G^{[d]}$ is asymptotically equal to

$$\frac{n!^{d-1}}{e^n}(1+o(1))^n \text{ as } n \to \infty,$$

otherwise it has no transversals.

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Question

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Conjecture (Rodney)

Every latin square contains a 2-plex.

Main results

Theorem (T., 2018+)

Let G be a binary quasigroup of order n.

1. For all odd d the d-iterated quasigroup $G^{[d]}$ has a k-multiplex. If for some even d' the quasigroup $G^{[d']}$ has a k-multiplex then quasigroups $G^{[d]}$ contain k-multiplexes for all $d \geq d'$.

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2. There exists a constant $c = c(G, k)$ such that if $G^{[d]}$ has a k-multiplex then for large d it has asymptotically $c \left(\frac{(kn)!}{k!^n} \right)^d$ k-multiplexes.

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Corollary (T., 2018+)

Let G be a binary quasigroup of order n.

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Corollary (T., 2018+)

For a given binary quasigroup G and large d, a typical k-multiplex in the d-iterated quasigroup $G^{[d]}$ is a k-plex that cannot be partitioned into smaller plexes.

Inception of transversals

0	3	1	2
3	0	2	1
2	1	0	3
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0	3	1	2
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0	3	1	2	1	2	0	3	3	1	2	0	2	0	3	1
3	0	2	1	2	1	3	0	1	3	0	2	0	2	1	3
2	1	0	3	3	0	1	2	0	2	3	1	1	3	2	0
1	2	3	0	0	3	2	1	2	0	1	3	3	1	0	2
3	0	2	1	2	1	3	0	0	2	1	3	1	3	0	2
0	3	1	2	1	2	0	3	2	0	3	**1**	3	1	2	0
1	2	3	0	0	3	2	1	3	1	0	2	2	0	1	3
2	1	0	3	3	0	1	2	1	3	2	0	0	2	3	1
1	2	3	0	3	0	1	2	2	3	0	1	0	1	2	3
2	1	0	3	0	3	2	1	3	2	1	0	1	0	3	2
0	3	1	2	2	1	3	0	1	0	2	3	3	2	0	1
3	0	2	1	1	**2**	0	3	0	1	3	2	2	3	1	0
2	1	0	3	0	3	2	1	1	0	3	2	3	2	1	0
1	2	3	0	3	0	1	2	0	1	2	3	2	3	0	1
3	0	2	1	1	2	0	3	2	3	1	0	0	1	**3**	2
0	3	1	2	2	1	3	0	3	2	0	1	1	0	2	3

Proof method for transversals

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Given a multiset U of size n over a quasigroup G, denote by $t_U(d)$ the number of diagonals $I = (\alpha^1, \ldots, \alpha^n)$ in the Cayley table Q of $G^{[d]}$ such that $U = \{q_{\alpha^1}, \ldots, q_{\alpha^n}\}$.

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$$t_U(d) = \sum_V b_{U,V} t_V(d-1),$$

where $b_{U,V}$ is the number of permutations W on G for which

$$U = V * W.$$

Proof method for transversals

$$\mathbb{Z}_5^{[1]} = \ 0 \ \ 1 \ \ 2 \ \ 3 \ \ 4$$

$$t_{01234}(1) = 1; \qquad t_U(1) = 0 \text{ for all other } U.$$

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$$\mathbb{Z}_5^{[1]} = \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix}$$

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$$\mathbb{Z}_5^{[2]} = \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{matrix}$$

$$t_{01234}(2) = 15 = 15 \cdot t_{01234}(1).$$

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$$t_{00000}(2) = \ldots = t_{44444}(2) = 1 = 1 \cdot t_{01234}(1).$$

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$$t_{00014}(2) = t_{00023}(2) = \ldots = t_{44403}(2) = t_{44412}(2) = 5 = 5 \cdot t_{01234}(1).$$

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$$t_{00113}(2) = t_{11224}(2) = t_{22330}(2) = t_{33441}(2) = t_{00442}(2) = 10 = 10 \cdot t_{01234}(1).$$

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$$t_{01234}(2) = 15 \cdot t_{01234}(1);$$

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$$t_U(2) = 0 \text{ for all other } U.$$

Proof method for transversals

$$\mathbb{Z}_5^{[1]} = \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix}$$

$$y_1(1) = t_{01234}(1) = 1; \qquad y_2(1) = y_3(1) = y_4(1) = 0.$$

$$\mathbb{Z}_5^{[2]} = \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{matrix}$$

$$y_1(2) = t_{01234}(2) = 15;$$
$$y_2(2) = t_{00000}(2) + \ldots + t_{44444}(2) = 5;$$
$$y_3(2) = t_{00014}(2) + t_{00023}(2) + \ldots + t_{44403}(2) + t_{44412}(2) = 50;$$
$$y_4(2) = t_{00113}(2) + t_{11224}(2) + t_{22330}(2) + t_{33441}(2) + t_{00442}(2) = 50.$$

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$$y_1(2) = 15 \cdot y_1(1);$$
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$$y_2(2) = 5 \cdot y_1(1);$$
$$y_3(2) = 50 \cdot y_1(1);$$
$$y_4(2) = 50 \cdot y_1(1).$$

Do the same for transversals in the 3-iterated group $\mathbb{Z}_5^{[3]} \ldots$

Proof method for transversals

$$y_1(2) = 15 \cdot y_1(1);$$
$$y_2(2) = 5 \cdot y_1(1);$$
$$y_3(2) = 50 \cdot y_1(1);$$
$$y_4(2) = 50 \cdot y_1(1).$$

Do the same for transversals in the 3-iterated group $\mathbb{Z}_5^{[3]} \dots$

$$y_1(3) = 15 \cdot y_1(2) + 120 \cdot y_2(2) + 30 \cdot y_3(2) + 20 \cdot y_4(2);$$
$$y_2(3) = 5 \cdot y_1(2);$$
$$y_3(3) = 50 \cdot y_1(2) + 30 \cdot y_3(2) + 40 \cdot y_4(2);$$
$$y_4(3) = 50 \cdot y_1(2) + 60 \cdot y_3(2) + 60 \cdot y_4(2).$$

Proof method for transversals

$$\begin{pmatrix} y_1(d) \\ y_2(d) \\ y_3(d) \\ y_4(d) \end{pmatrix} = \begin{pmatrix} 15 & 120 & 30 & 20 \\ 5 & 0 & 0 & 0 \\ 50 & 0 & 30 & 40 \\ 50 & 0 & 60 & 60 \end{pmatrix} \begin{pmatrix} y_1(d-1) \\ y_2(d-1) \\ y_3(d-1) \\ y_4(d-1) \end{pmatrix};$$

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$$Y(d) = A(\mathbb{Z}_5)Y(d-1).$$

Proof method for transversals

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$$Y(d) = A(\mathbb{Z}_5)Y(d-1).$$

The number of transversals $T(d)$ in the d-iterated group $\mathbb{Z}_5^{[d]}$ is the first component of the vector $Y(d)$ obtained as the result of process

$$Y(d) = A(\mathbb{Z}_5)Y(d-1),$$

where matrix $A(\mathbb{Z}_5)$ does not depend on d and is derived from only the group $\mathbb{Z}_5$.

Markov processes and ergodic theorem

Lemma

For every quasigroup G of order n the matrix $A(G)$ has column sums $n!$ and (after normalization) defines an irreducible Markov process with period at most 2.

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A Markov process with transition matrix A is irreducible and aperiodic if and only if it converges to the stationary distribution with all nonzero components.

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For every quasigroup G of order n the matrix $A(G)$ has column sums $n!$ and (after normalization) defines an irreducible Markov process with period at most 2.

Ergodic theorem

A Markov process with transition matrix A is irreducible and aperiodic if and only if it converges to the stationary distribution with all nonzero components.

The stationary distribution for an ergodic process with a matrix A is the eigenvector of A for the largest eigenvalue.

Proof method for transversals

$$Y(d) = A(\mathbb{Z}_5)Y(d-1);$$

$$A(\mathbb{Z}_5) = \begin{pmatrix} 15 & 120 & 30 & 20 \\ 5 & 0 & 0 & 0 \\ 50 & 0 & 30 & 40 \\ 50 & 0 & 60 & 60 \end{pmatrix}.$$

The process defined by $A(\mathbb{Z}_5)$ is aperiodic.

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The process defined by $A(\mathbb{Z}_5)$ is aperiodic.

The number of transversals $T(d)$ in the d-iterated group $\mathbb{Z}_5^{[d]}$ converges to $c \cdot 120^{d-1}$, where $c > 0$ is the first component of the normalized eigenvector $1/125(24, 1, 40, 60)^T$ for the largest eigenvalue $\lambda = 120$ of the matrix $A(\mathbb{Z}_5)$.

Proof method for transversals

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$$T(d) = \frac{24}{125} \cdot 120^{d-1}(1 + o(1)).$$

Number of transversals in d-iterated groups and quasigroups of small order

$G \setminus d$	even	odd
$\mathbb{Z}_2$	0	2^{d-1}
$\mathbb{Z}_3$	$2/3 \cdot 6^{d-1} - 3^{d-2}$	$2/3 \cdot 6^{d-1} + 3^{d-2}$
$\mathbb{Z}_4$	0	$3/8 \cdot 24^{d-1} + 5 \cdot 8^{d-2}$
$\mathbb{Z}_2^2$	$3/8 \cdot 24^{d-1} - 8^{d-2}$	$3/8 \cdot 24^{d-1} + 5 \cdot 8^{d-2}$
G_4	$3/32 \cdot 24^{d-1}(1 + o(1))$	
$\mathbb{Z}_5$	$24/125 \cdot 120^{d-1}(1 + o(1))$	
G_5	$24/625 \cdot 120^{d-1}(1 + o(1))$	

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$\mathbb{Z}_2$	0	2^{d-1}
$\mathbb{Z}_3$	${}^{2}/_{3} \cdot 6^{d-1} - 3^{d-2}$	${}^{2}/_{3} \cdot 6^{d-1} + 3^{d-2}$
$\mathbb{Z}_4$	0	${}^{3}/_{8} \cdot 24^{d-1} + 5 \cdot 8^{d-2}$
$\mathbb{Z}_2^2$	${}^{3}/_{8} \cdot 24^{d-1} - 8^{d-2}$	${}^{3}/_{8} \cdot 24^{d-1} + 5 \cdot 8^{d-2}$
G_4	${}^{3}/_{32} \cdot 24^{d-1}(1 + o(1))$	
$\mathbb{Z}_5$	${}^{24}/_{125} \cdot 120^{d-1}(1 + o(1))$	
G_5	${}^{24}/_{625} \cdot 120^{d-1}(1 + o(1))$	

$$G_4^2 = \begin{matrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 \\ 2 & 1 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{matrix}$$

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$\mathbb{Z}_2$	0	2^{d-1}
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G_5	$24/625 \cdot 120^{d-1}(1 + o(1))$	

$$G_4^2 = \begin{matrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 \\ 2 & 1 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{matrix} \qquad G_5^2 = \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 1 & 2 & 0 \\ 4 & 2 & 0 & 1 & 3 \end{matrix}$$

Open questions

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2. How are the constants $c(G, k)$ for a given G related to each other for different k?

3. Does the d-iterated group $\mathbb{Z}_n^{[d]}$ have the maximum number of transversals (or k-plexes and k-multiplexes) among all d-iterated quasigroups of order n (if it is nonzero)?

Thank you for your attention!